Assignment 6

Deadline: March 9, 2018

Hand in: Supp. Ex no 1, 2, 5, and 9.

Section 7.2: No 18, 19.

Supplementary Exercise

1. Let $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = -\min\{f(x), 0\}$. Show that f_+ and f_- are both integrable when f is integrable on $[a, b]$.

2. Let g be differed from f by finitely many points. Show that g is integrable if f is integrable over $[a, b]$ and they have the same integral over $[a, b]$.

3. Let f be non-negative and continuous on $[a, b]$. Show that $\int_a^b f = 0$ if and only if $f \equiv 0$.

4. Order the rational numbers in $[0, 1]$ into a sequence $\{z_n\}$ and define
$$\varphi(x) = \sum_{\{j,\ z_j<x\}} \frac{1}{2^j} .$$
Show that φ is continuous at every irrational number but discontinuous at every rational number in $(0, 1)$. Is it integrable?

5. Display two integrable functions f and Φ so that $\Phi \circ f$ is not integrable. Hint: Take f to be the Thomae's function.

6. Let $f \in \mathcal{R}[a, b]$ and $g \in C^1[c, d]$ where $f[a, b] \subset [c, d]$. Show that the composite $g \circ f \in \mathcal{R}[a, b]$. Here C^1 means continuously differentiable.

7. (Optional). Let $f \in \mathcal{R}[a, b]$ and $g \in C[c, d]$ where $f[a, b] \subset [c, d]$. Show that the composite $g \circ f \in \mathcal{R}[a, b]$. Hint: For $\varepsilon > 0$, fix δ_0 such that $|g(z_1) - g(z_2)| < \varepsilon$ for $|z_1 - z_2| < \delta_0$. For $\varepsilon,\ \delta_0 > 0$, there exists a partition P such that $\sum_j osc_{I_j} f \Delta x_j < \varepsilon\delta_0$. Then apply the Second Criterion.

8. Let f be a continuous function on $[a, b]$ and g a nonnegative integrable function on the same interval. Prove the mean-value theorem for integral:
$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx,$$
for some $c \in [a, b]$.

9. Evaluate the following limits:

   (a)
$$\lim_{n\to\infty} \Big(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\Big);$$

(b)

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}.$$

10. (Optional).

    (a) Establish the Cauchy-Schwarz Inequality in integral form: For integrable f and g on $[a, b]$,

    $$\int_a^b |fg| \leq \sqrt{\int_a^b f^2}\sqrt{\int_a^b g^2}.$$

    (b) Deduce the following Cauchy-Schwarz Inequality for vectors

    $$\sum_{k=1}^n |a_k b_k| \leq \sqrt{\sum_{k=1}^n a_k^2}\sqrt{\sum_{k=1}^n b_k^2}.$$

11. (Optional.) Let J be a convex function on some $[-M-1, M+1]$ and $f \in R[0,1]$ satisfying $|f(x)| \leq M$. Establish Jensen's Inequality in integral form

    $$J\left(\int_0^1 f(x)dx\right) \leq \int_0^1 J(f(x))dx\ .$$